ON INTEGRABLE IN QUADRATURES EVOLUTION PARTIAL SUPER-DIFFERENTIAL EQUATIONS ON SUPERSPACES

Anatolij K. Prykarpatsky
Lviv Polytechnic National University, Lviv, Ukraine
e-mail: pryk.anat@cybergal.com

Let $\mathbb{R}^{1|1} \simeq \mathbb{R} \times \Lambda^{(1|1)}$ denote the superized real axis $\mathbb{R}$ by means of the one-dimensional $\mathbb{Z}_2$-graded Grassmann algebra $\Lambda^{1|1} = \Lambda_0^{1|1} \oplus \Lambda_1^{1|1}$ with coordinates $(x,\theta) \in \mathbb{R} \times \Lambda^{1|1}$ and $D_\theta := \partial/\partial\theta + \theta\partial/\partial x$ be the correspondinf super-derivation, acting on the space $C^\infty(\mathbb{R}^{1|1};\Lambda^{1|1})$ of smooth functions on $\mathbb{R}^{1|1}$. For any smooth function $u^{(p)} \in C^\infty(\mathbb{R}^{1|1};\Lambda^{1|1})$ of parity $(p) \in \mathbb{Z}_2$ there exists the expansion

$$(1) \qquad u^{(p)}(x,\theta) = u^{(p)}(x) + \theta u^{(p+1)}(x)$$

at any point $(x,\theta) \in \mathbb{R}^{1|1}$ with uniquely defined smooth mappings $u^{(k)} \in C^\infty(\mathbb{R}^{1|1};\Lambda^{1|1}_{(k)})$, $k = p, p+1$. For the function [1, 2] there is defined the super-integral $\int u^{(p)}(x,\theta)d\theta, (p) \in \mathbb{Z}_2$, over the super-variable $\theta \in \Lambda_1^{1|1}$ via the rules:

$$(2) \qquad \int 1 d\theta = 0, \int \theta d\theta = 1.$$

As a simplest example, a general linear second-order non-uniform and non-autonomous evolution partial super-differential equation looks as

Similarly, one can write down a nonlinear autonomous second-order evolution partial super-differential equation

$$(4) \qquad \partial u/\partial t = K(x,\theta; u, D_\theta u, D_\theta^2 u),$$

where the mapping $K : \mathbb{R}^{1|1} \times \mathfrak{M} \to T(\mathfrak{M})$ above is interpreted as a vector super-field on the functional jet-supermanifold $\mathfrak{M} \simeq J^2(\mathbb{R}^{1|1};\Lambda_1^{(1|1)})$, pameterized by points $(x,\theta) \in \mathbb{R}^{1|1}$ and $t \in \mathbb{R}$.

Definition 1. *We will call the evolution super-differential equations like (5) or (4) integrable by quadratures, if these equations possess analytical functional invariants, that is conservation laws* $\gamma \in D(\mathbb{R}^{1|1} \times \mathfrak{M})$, *invariant with respect to the evolution parameter* $t \in \mathbb{R}$:

$$d\gamma/dt = \partial\gamma/\partial t + (\operatorname{grad}\gamma|K) = 0, \tag{5}$$

where $\varphi := \operatorname{grad}\gamma \in T^{*}(\mathfrak{M})$ *denotes the usual super-gradient of the functional* $\gamma \in D(\mathbb{R}^{1|1} \times \mathfrak{M})$ *with respect to the mapping* $u \in C^{2}(\mathbb{R}^{1|1}; \Lambda_1^{1|1})$.

The following generalization of the classical Noether-Lax lemma makes it possible to describe invariants of the vector super-field (4)

Lemma 1. ***(Noether-Lax type generalization)*** *Let* $\gamma \in D(\mathbb{R}^{1|1} \times \mathfrak{M})$ *be an invariant of the vector super-field (4) on the supermanifold* $\mathfrak{M}$ *and* $\varphi = \operatorname{grad}\gamma \in T^{*}(\mathfrak{M})$ *be its gradient. Then the following functional super-differential evolution equation*

$$\partial\varphi/\partial t + K^{\prime,*}\varphi = 0 \tag{6}$$

holds on the whole super-space $\mathbb{R}^{1|1} \times \mathfrak{M}$ *for all* $t \in \mathbb{R}$, *where* $K^{\prime,*} : T^{*}(\mathfrak{M}) \to T^{*}(\mathfrak{M})$ *is the adjoint operator to the Frechet derivative* $K' : T(\mathfrak{M}) \to T(\mathfrak{M})$ *of the vector super-field (4) with respect to the natural bilinear super-form* $(\cdot|\cdot) : T^{*}(\mathfrak{M}) \times T(\mathfrak{M}) \to \Lambda^{1|1}$ *on the Euclidean product* $T^{*}(\mathfrak{M}) \times T(\mathfrak{M})$.

Our work is devoted to a development of the classical Hormander [1] approach to studyng the asymptotic solutions to the Noether-Lax type evolution equation (6) and their application for constructing invariants [2] of nonlinear evolution parttal super-differential equations (4).

References

[1] Hörmander L., Linear Partial Differential OperatorsSpringer Berlin, Heidelberg, 1964.

[2] Blackmore D., Prykarpatsky A.K. and Samoylenko V.H., Nonlinear dynamical systems of mathematical physics, World Scientific Publisher, NJ, USA, 2011